① Arithmetic-Geometric correspondence

$\overline{\mathbb{F}}_q$	$\mathbb{C}$	(Lefschetz principle)
variety $X/\mathbb{F}_q$	variety $X/\mathbb{C}$	
$H^k_c(\overline{X}_{ét}, \mathbb{Q}_\ell) \circlearrowleft$ $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-repr.	$H^k_c(X_{an}, \mathbb{Q}) \circlearrowleft$ MHS	(Weil conjectures) Deligne
weight filtr.	weight filt.	
$\mathrm{tr}(F_{q^m}, H^k_c(\overline{X}_{ét}, \mathbb{Q}_\ell))$	$Gr^p_F Gr^W_{p+q} H^k_c(X_{an}, \mathbb{C})$	
$\underbrace{\sum_{k\geq 0} (-1)^k \mathrm{tr}(-)}_{}$	$\underbrace{\sum_{\substack{k\geq 0\\ p,q}} (-1)^k \dim(-) u^p v^q \in \mathbb{Z}[u,v]}_{}$	
$= \#X(\mathbb{F}_{q^m})$ by Grothendieck-Lefschetz trace formula	$=: e(X)$	"E-polynomial" / "Serre polynomial"

<u>Thm</u> (Katz) If X and Y are varieties / $\mathbb{Z}$ (→ fin. gen. $\mathbb{Z}$-alg.)
such that $\#X(\mathbb{F}_q) = \#Y(\mathbb{F}_q)$ for all q. (→ and $R \to \mathbb{F}_q$)
then $e(X_\mathbb{C}) = e(Y_\mathbb{C})$.

<u>Remark</u>
- $e(X) = e(\underset{\substack{\text{closed}\\ \text{subvar.}}}{Z} \sqcup X\setminus Z) = e(Z) + e(X\setminus Z)$
- If $\#X(\mathbb{F}_q) = \sum_n a_n q^n$

  then $e(X_\mathbb{C}) = \sum_n a_n e(\mathbb{A}^n_\mathbb{C}) = \sum_n a_n (uv)^n$
- If $P \to X$ is a G-torsor (étale)

  ↳ connected

  then $e(P) = e(G\times X) = e(G)e(X)$

② Character stacks

G = alg. group (typically GL_n, SL_n)

M = smooth manifold, compact, connected

<u>G-character stack of M</u> $\mathfrak{X}_G(M)$ = moduli space of $\underbrace{\text{G-local systems on } M}_{\pi_1(M) \longrightarrow G}$

$\mathrm{Rep}_G(M) := \mathrm{Hom}(\pi_1(M), G)$ <u>G-repr. variety</u>.

$\mathfrak{X}_G(M) := [\,\mathrm{Rep}_G(M) / G\,]$

For $M = \Sigma_g$ (closed orient. surface genus g):

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1,b_1] \cdots [a_g, b_g] = 1 \rangle$$

$$\mathrm{Rep}_G(\Sigma_g) = \{ (A_1, B_1, \dots, A_g, B_g) \in G^{2g} \mid [A_1, B_1] \cdots [A_g, B_g] = 1 \} \subseteq G^{2g}$$

<u>Goal</u>: Study MHS, or E-polynomial, of $\mathfrak{X}_G(\Sigma_g)$ and $\mathrm{Rep}_G(\Sigma_g)$.

<u>Thm</u> (Frobenius)

$$\# \mathrm{Rep}_G(\Sigma_g)(\mathbb{F}_q) = \# G(\mathbb{F}_q) \cdot \sum_{\chi \in \widehat{G(\mathbb{F}_q)}} \left(\frac{\# G(\mathbb{F}_q)}{\chi(1)} \right)^{2g-2}$$

used by Hausel; Rodriguez-Villegas to compute e-polyn. of GL_n - repr. varieties

③ Counting $\mathbb{F}_q$-points of $\mathrm{Rep}_G(\Sigma_g)$

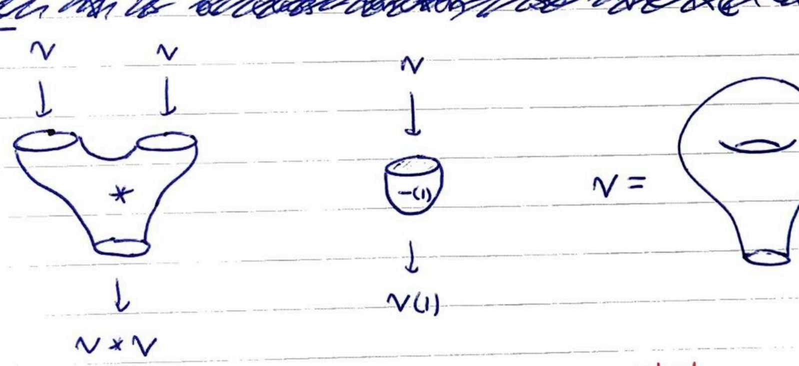
$$\#\{(A_1, B_1, \dots, A_g, B_g) \in G(\mathbb{F}_q)^{2g} \,/\, [A_1, B_1]\cdots[A_g, B_g] = 1\}$$

- Let $\nu \in \mathbb{Z}^{G(\mathbb{F}_q)}$ be $\nu(x) = \#\{(A,B) \in G(\mathbb{F}_q)^2 \mid [A,B] = x\}$

- So, $\#\mathrm{Rep}_G(\Sigma_1)(\mathbb{F}_q) = \nu(1)$.

- $$\#\mathrm{Rep}_G(\Sigma_2)(\mathbb{F}_q) = \sum_{xy = \underset{z}{1}} \nu(x)\nu(y) = (\nu * \nu)\overset{z}{(1)}$$

- Generally, $\#\mathrm{Rep}_G(\Sigma_g)(\mathbb{F}_q) = (\underbrace{\nu * \cdots * \nu}_{g \text{ times}})(1)$



Construction can be upgraded to a $\overset{\text{monoidal}}{\vee}$ functor (TQFT)

$$Z : \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}$$

$$S^1 \longmapsto \cancel{\mathbb{Q}^{G(\mathbb{F}_q)}}$$

$$R_{\mathbb{C}}(G(\mathbb{F}_q))$$

(4) Connection to geometry ($/\mathbb{C}$)

Similar construction by Gonzalez-Prieto, Logares, Muñoz
(used to compute E-polynomials of $SL_2(\mathbb{C})$-repr. variety)

$$Z : \mathrm{Bord}_2 \longrightarrow \text{"Vect"} \quad K_0(MHS)\text{-Mod}$$

$$S^1 \longmapsto K_0(MHM/G)$$

Convolution:

$$K_0(MHM/G) \times K_0(MHM/G) \longrightarrow K_0(MHM/G)$$

$$(\mathcal{F}, \mathcal{G}) \longmapsto m_!(\pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G})$$

$$\begin{tikzcd} G & & G \\ & G \times G \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[d, "m"] & \\ & G & \end{tikzcd}$$

Evaluation in 1:

$$K_0(MHM/G) \longrightarrow K_0(MHS)$$

$$\mathcal{F} \longmapsto i^* \mathcal{F}$$

$$\begin{tikzcd} G \\ \{1\} \arrow[u, "i"'] \end{tikzcd}$$

<u>Remark</u>
- Construction works in any dimension n
- Can replace $K_0(MHS)$ by $K_0(Var)$

⑤ Our work

- Reformulated the arithmetic (point-count.) method in terms of TQFT $Z^{\text{arith.}}_{G(\mathbb{F}_q)}$.
- Generalized $Z^{\text{arithm.}}_{G(\mathbb{F}_q)}$ to any dim. n.
- Correspondence between arithm. & geom. TQFTs:

$$
\begin{tikzcd}
 & \mathrm{Vect}_{\mathbb{Q}} \arrow[d, "\#^*"] \\
\mathrm{Bord}_n \arrow[ur, bend left, "Z^{\text{arith.}}_{G(\mathbb{F}_q)}"] \arrow[dr, bend right, "Z^{\text{geom.}}_{G}"'] & \\
 & K_0(\mathrm{Var})\text{-Mod.}
\end{tikzcd}
\qquad
\begin{array}{c}
\Uparrow \text{ if } G \text{ connected}
\end{array}
\qquad
K_0(\mathrm{Var}) \xrightarrow{\#} \mathbb{Z} \subseteq \mathbb{Q}
$$